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Majorana representations of A_5

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Abstract The Monster group M, which is the largest among the 26 sporadic simple groups is the automorphism group of the 196,884-dimensional Conway–Griess–Norton algebra (simply called the Monster algebra). There is a remarkable correspondence between the so-called $2A$-involutions in M and certain idempotents in the Monster algebra (we refer to these idempotents as Majorana axes). The isomorphism types of the subalgebras in the Monster algebra generated by pairs of Majorana axes were calculated by S. Norton a while ago (there are precisely nine isomorphism types). More recently these nine algebras were characterized by S. Sakuma in the context of Vertex Operator Algebras, relying on earlier work by M. Miyamoto. The properties of Monster algebras used in the proof of Sakuma's theorem are rather elementary and they have been axiomatized under the name of Majorana representations. In this terminology Sakuma's theorem amounts to classification of the Majorana representations of the dihedral groups together with a remark that all the representations are based on embeddings into the Monster. In the present paper it is shown that the alternating group A_5 of degree 5 possesses precisely two Majorana representations, both based on embeddings into the Monster. The dimensions of the representations are 20 and 26; the scalar squares of their identities are 10 and $72/7$, respectively (in the Vertex Operator Algebra context these numbers are doubled central charges).

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1 The Monster group and its algebra

The Monster group M contains two conjugacy classes of involutions with representatives t and z, so that

$$C_M(t) \cong 2 \cdot BM \quad \text{and} \quad C_M(z) \cong 2_+^{1+24}.Co_1,$$

where BM is the Baby Monster sporadic simple group, Co_1 is the largest Conway sporadic simple group, whose double cover is the automorphism group of the Leech lattice Λ, $Q := O_2(C_M(z)) \cong 2_+^{1+24}$ is an extraspecial group with $Q/\langle z\rangle$ and $\Lambda/2\Lambda$ being isomorphic modules for $C_M(z)/Q \cong Co_1$. Thus the structure of $C_M(z)$ is well understood independently of the Monster. The M-conjugates of t and z are called $2A$- and $2B$-involutions, respectively (this is because $|C_M(t)| > |C_M(z)|$).

One can say that the Monster was discovered through t and constructed through z. The $2A$-involutions form a class of 6-transpositions in the sense that their pairwise products have orders at most 6. The Baby Monster is a $\{3, 4\}$-transposition group and its plays an intermediate role between the Monster and Fischer's 3-transposition groups. The M-orbit containing a pair of $2A$-involutions is uniquely determined by the conjugacy class containing their product, while the totality of these products constitutes the union of the following nine conjugacy classes of M:

$$1A,\ 2A,\ 2B,\ 3A,\ 3C,\ 4A,\ 4B,\ 5A,\ 6A.$$

Thus there are precisely nine M-orbits on the pairs of $2A$-involutions and these orbits are naturally indexed by the above conjugacy classes.

The Monster contains a $2B$-pure subgroup Z_2 of order 2^2 such that

$$N_M(Z_2) \cong 2^{2+11+22}.(M_{24} \times S_3),$$

where M_{24} is the largest Mathieu sporadic simple group. Similarly to $C_M(z)$ the subgroup $N_M(Z_2)$ can be constructed outside the Monster, for instance in terms of Parker's loop or tri-extraspecial groups (cf. [4] or Chapter 2 in [5]).

It can be shown (cf. [4]) that the degree of a non-trivial characteristic zero representation of the free amalgamated product of $C_M(z)$ and $N_M(Z_2)$ over their intersection is at least 196,883 and on the early stages of studying the Monster it was conjectured that the Monster does possess a faithful module Π_1 over the real numbers of that dimension. Then it was observed by S. Norton that (up to scalars) Π_1 must carry a unique inner product $(\ ,\ )_1$, an algebra product $\cdot_1$ and a trilinear form, the latter can be taken to be

$$(u, v, w) \mapsto (u \cdot_1 v, w)_1,$$

which are M-invariant.

Thompson [15] has proved that (up to conjugation) there is at most one non-trivial homomorphism ϑ_1 of the free amalgamated product of $C_M(z)$ and $N_M(Z_2)$ (over their intersection) into $GL(\Pi_1)$. In these terms Griess's construction of the Monster in [3] amounts to the existence proof for ϑ_1 followed by a justification that $\vartheta_1(C_M(z))$ is the full centralizer of $\vartheta_1(z)$ in the image of ϑ_1. In both stages the Norton algebra has played an essential role.

The final stretch in the uniqueness proof for the Monster was made by Norton [10] by deducing the existence of Π_1 from the local structure of the Monster. Norton has computed the parameters of the symmetric rank 9 association scheme corresponding to the action of M by conjugation on the class of its $2A$-involutions. The general theory of association schemes applied to these parameters shows that Π_1 appears in the irreducible decomposition of the corresponding permutation module with multiplicity 1. By the Frobenious reciprocity this means that $C_M(t)$ has 1-dimensional centralizer in Π_1 (the restriction of the Norton algebra to this centralizer happens to be non-trivial).

Conway, when revised in [1] the construction of the Monster by R. Griess, has adjoined to Π_1 a 1-dimensional trivial direct summand to obtain a 196,884-dimensional M-module Π on which he has defined what is now known as the Conway–Griess–Norton algebra $\cdot$ and an associated inner product $(\ ,\ )$. Since

$$\vartheta : M \to GL(\Pi)$$

is the direct sum of ϑ_1 with the trivial 1-dimensional representation, the spaces of M-invariant inner and algebra products on Π are both 2-dimensional. The particular choice made by Conway was dictated by the following principle. Let S be the largest solvable normal subgroup in $N_M(Z_2)$, so that $N_M(Z_2)/S \cong M_{24}$. Then $C_\Pi(S)$ is isomorphic to the 24-dimensional permutational module of the latter factor group (cf. Proposition 3.1.21 (vii) in [5]) and the restrictions of $\cdot$ and $(\ ,\ )$ to $C_\Pi(S)$ are the natural point-wise (known as the Hadamard) products. It is most remarkable that this is precisely the choice dictated by the vertex operator algebra structure on the Moonshine modules for which $(\Pi, (\ ,\ ), \cdot)$ is the Griess algebra.

The triple $(\Pi, (\ ,\ ), \cdot)$ was proved in [1] to satisfy the following (with Π in the place of V):

(M1) $(\ ,\ )$ is a symmetric positive definite bilinear form on V that associates with $\cdot$ in the sense that $(u, v \cdot w) = (u \cdot v, w)$ for all $u, v, w \in V$, and $\cdot$ is a bilinear commutative non-associative algebra product on V;
(M2) the Norton inequality holds, so that $(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$ for all $u, v \in V$.

The trivial summand of Π is linearly spanned by the identity ι of $(\Pi, \cdot)$; the centralizer of $C_M(t)$ in Π is 2-dimensional containing four $\cdot$-idempotents, which are 0, ι, a_t and $\iota - a_t$, where a_t is a *Majorana axis* in the sense that the following conditions hold (with a_t in place of a and Π in place of V):

(M3) $(a, a) = 1$ and $a \cdot a = a$, so that a is a $\cdot$-idempotent of length 1;
(M4) $V = V_1^{(a)} \oplus V_0^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)} \oplus V_{\frac{1}{2^5}}^{(a)}$, where $V_\mu^{(a)} = \{v \mid v \in V, a \cdot v = \mu v\}$ is the set of μ-eigenvectors of (the adjoint action of) a on V;
(M5) $V_1^{(a)} = \{\lambda a \mid \lambda \in \mathbf{R}\}$;
(M6) the linear transformation $\tau(a)$ of V defined via

$$\tau(a) : u \mapsto (-1)^{2^5 \mu} u$$

for $u \in V_\mu^{(a)}$ with $\mu = 1, 0, \frac{1}{2^2}, \frac{1}{2^5}$, preserves the algebra product (*i.e.* $u^{\tau(a)} \cdot v^{\tau(a)} = (u \cdot v)^{\tau(a)}$ for all $u, v \in V$);
(M7) if $V_+^{(a)}$ is the centralizer of $\tau(a)$ in V, so that $V_+^{(a)} = V_1^{(a)} \oplus V_0^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)}$, then the linear transformation $\sigma(a)$ of $V_+^{(a)}$ defined via

$$\sigma(a) : u \mapsto (-1)^{2^2 \mu} u$$

for $u \in V_\mu^{(a)}$ with $\mu = 1, 0, \frac{1}{2^2}$ preserves the restriction of the algebra product to $V_+^{(a)}$ (*i.e.* $u^{\sigma(a)} \cdot v^{\sigma(a)} = (u \cdot v)^{\sigma(a)}$ for all $u, v \in V_+^{(a)}$).

In the case of the Monster algebra $\tau(a_t) = \vartheta(t)$.

The conditions (M1)–(M7) imply that the eigenspaces $V_\mu^{(a)}$ of (the adjoint action of) a satisfy the fusion rules described in Table 1 (where $Sp = \{1, 0, \frac{1}{2^2}, \frac{1}{2^5}\}$ is the spectrum of a).

Table 1 The fusion rules

Sp	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
1	1	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
0	0	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
$\frac{1}{2^2}$	$\frac{1}{2^2}$	$\frac{1}{2^2}$	$1, 0$	$\frac{1}{2^5}$
$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$\frac{1}{2^5}$	$1, 0, \frac{1}{2^2}$

The meaning of the fusion rules is the inclusion

$$V_\lambda^{(a)} \cdot V_\mu^{(a)} \subseteq \bigoplus_{\nu \in Sp(\lambda,\mu)} V_\nu^{(a)}$$

where $\lambda, \mu \in Sp$ and $Sp(\mu, \lambda)$ is the (λ, μ)-entry in Table 1.

The Majorana properties of the idempotents a_t are (implicitly) stated in [11], while in [8] they are proved in the Vertex Operator Algebra context. A self-contained proof based on (a modified) Griess's construction is given in [5] (cf. Proposition 8.6.2).

The subalgebras in the Monster algebra generated by pairs of Majorana axes appeared already in [2]. The structure of some of the nine algebras were justified in [1] and of the remaining ones in [11]. These algebras are given in Table 2 whose content is explained below.

Let t_0 and t_1 be $2A$-involutions in M, let $a_0 = a_{t_0}$ and $a_1 = a_{t_1}$ be the corresponding Majorana axes, and let $\rho = t_0 t_1$. For $\varepsilon \in \{0, 1\}$ let $a_{\varepsilon+2i}$ be the image of a_ε under the ith power of ρ (alternatively $a_{\varepsilon+2i}$ can be defined as the Majorana axis associated with $\rho^{-i} t_\varepsilon \rho^i$). Then the M-conjugacy class of ρ gives the name to the subalgebra in the Monster algebra generated by a_0 and a_1 (the leftmost column of Table 2). In the $1A$-type we have $a_0 = a_1$, the algebra is 1-dimensional and this case is excluded from the table. In the $2A$-type ρ is a $2A$-involution and later on we will axiomatize this property as (M8) below. In the subalgebras of type $3A$, $4A$ or $5A$, the 1-dimensional subspace spanned by the vector u_ρ, v_ρ or w_ρ is invariant under the normalizer $N_M(\langle\rho\rangle)$ isomorphic to $3 \cdot F_{24}$, $2_+^{1+24}.Co_3$ or $(D_{10} \times F_5).2$, respectively. Furthermore, in the types $3A$ and $4A$ the vector itself is stable under $N_M(\langle\rho\rangle)$, while in the type $5A$ it is preserved up to negation and satisfies the following:

$$w_\rho = -w_{\rho^2} = -w_{\rho^3} = w_{\rho^4}.$$

Thus Aut $(5A)$ contains a Frobenius subgroup of order 20 acting naturally on $\{a_{-2}, a_{-1}, a_0, a_1, a_2\}$ with a D_{10}-subgroup centralizing w_ρ and the remaining elements negating this vector. Notice that Norton's inner product in [11] is 16 times ours, and his t_0, u, v, and w are 64, 90, 192, and 8,192 times our a_0, u_ρ, v_ρ, and w_ρ, respectively.

The two-Majorana-generated subalgebras in the Monster algebra were characterized in the most remarkable paper by Sakuma [13].

Proposition 1.1 *Let V be a real vector space endowed with an inner product $(\ ,\ )$ and an algebra product $\cdot$, satisfying* (M1)*, and let a_0 and a_1 be a pair of Majorana axes in V (i.e., vectors satisfying the conditions* (M3)–(M7)*). Then either $a_0 = a_1$, or the subalgebra generated by a_0 and a_1 is isomorphic to one of the eight algebras in Table 2. In particular, the Norton inequality* (M2) *holds in the subalgebra.*

Originally Sakuma's theorem was stated in the context of vertex operator algebras, and Norton's inequality has been used in [13] via application of Theorem 9.1 from [9]. It was

Table 2 The two-Majorana-generated algebras

Type	Basis	Products and angles
2A	a_0, a_1, a_ρ	$a_0 \cdot a_1 = \frac{1}{2^3}(a_0 + a_1 - a_\rho), a_0 \cdot a_\rho = \frac{1}{2^3}(a_0 + a_\rho - a_1)$ $(a_0, a_1) = (a_0, a_\rho) = (a_1, a_\rho) = \frac{1}{2^3}$
2B	a_0, a_1	$a_0 \cdot a_1 = 0, (a_0, a_1) = 0$
3A	$a_{-1}, a_0, a_1,$ u_ρ	$a_0 \cdot a_1 = \frac{1}{2^5}(2a_0 + 2a_1 + a_{-1}) - \frac{3^3 \cdot 5}{2^{11}} u_\rho$ $a_0 \cdot u_\rho = \frac{1}{3^2}(2a_0 - a_1 - a_{-1}) + \frac{5}{2^5} u_\rho$ $u_\rho \cdot u_\rho = u_\rho$ $(a_0, a_1) = \frac{13}{2^8}, (a_0, u_\rho) = \frac{1}{2^2}, (u_\rho, u_\rho) = \frac{2^3}{5}$
3C	a_{-1}, a_0, a_1	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1}), (a_0, a_1) = \frac{1}{2^6}$
4A	$a_{-1}, a_0, a_1,$ a_2, v_ρ	$a_0 \cdot a_1 = \frac{1}{2^6}(3a_0 + 3a_1 + a_2 + a_{-1} - 3v_\rho)$ $a_0 \cdot v_\rho = \frac{1}{2^4}(5a_0 - 2a_1 - a_2 - 2a_{-1} + 3v_\rho)$ $v_\rho \cdot v_\rho = v_\rho, a_0 \cdot a_2 = 0$ $(a_0, a_1) = \frac{1}{2^5}, (a_0, a_2) = 0, (a_0, v_\rho) = \frac{3}{2^3}, (v_\rho, v_\rho) = 2$
4B	$a_{-1}, a_0, a_1,$ a_2, a_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho^2})$ $a_0 \cdot a_2 = \frac{1}{2^3}(a_0 + a_2 - a_{\rho^2})$ $(a_0, a_1) = \frac{1}{2^6}, (a_0, a_2) = (a_0, a_{\rho^2}) = \frac{1}{2^3}$
5A	$a_{-2}, a_{-1}, a_0,$ a_1, a_2, w_ρ	$a_0 \cdot a_1 = \frac{1}{2^7}(3a_0 + 3a_1 - a_2 - a_{-1} - a_{-2}) + w_\rho$ $a_0 \cdot a_2 = \frac{1}{2^7}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$ $a_0 \cdot w_\rho = \frac{7}{2^{12}}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{2^5} w_\rho$ $w_\rho \cdot w_\rho = \frac{5^2 \cdot 7}{2^{19}}(a_{-2} + a_{-1} + a_0 + a_1 + a_2)$ $(a_0, a_1) = \frac{3}{2^7}, (a_0, w_\rho) = 0, (w_\rho, w_\rho) = \frac{5^3 \cdot 7}{2^{19}}$
6A	$a_{-2}, a_{-1}, a_0,$ a_1, a_2, a_3 a_{ρ^3}, u_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{2^6}(a_0 + a_1 - a_{-2} - a_{-1} - a_2 - a_3 + a_{\rho^3}) + \frac{3^2 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot a_2 = \frac{1}{2^5}(2a_0 + 2a_2 + a_{-2}) - \frac{3^3 \cdot 5}{2^{11}} u_{\rho^2}$ $a_0 \cdot u_{\rho^2} = \frac{1}{3^2}(2a_0 - a_2 - a_{-2}) + \frac{5}{2^5} u_{\rho^2}$ $a_0 \cdot a_3 = \frac{1}{2^3}(a_0 + a_3 - a_{\rho^3}), a_{\rho^3} \cdot u_{\rho^2} = 0, (a_{\rho^3}, u_{\rho^2}) = 0$ $(a_0, a_1) = \frac{5}{2^8}, (a_0, a_2) = \frac{13}{2^8}, (a_0, a_3) = \frac{1}{2^3}$

shown in [6] that for the case of algebras generated by two Majorana axes the Norton inequality is a consequence of the conditions (M1) and (M3)–(M7). Since then following events related to the Norton inequality took place:

(I) It was also observed in [6] that the fact that two Majorana axes generating a $2B$-algebra associate with any other element follows from the fusion rules. Originally this fact was proved in [1] using the Norton inequality, and it has played an essential role in the determination of the Majorana representation of S_4 of the shape $(2B, 3A)$. This observation makes the proof of the main result of [6] Norton inequality free.

(II) The Norton inequality has been used neither in the present paper, nor in [7].

(III) A Majorana representation of an elementary abelian group of order 9 extended by a fixed-point-free involuntary automorphism where every pair of distinct Majorana axes generate a $3C$-subalgebra is not based on an embedding into the Monster (the Monster does not have $3C$-pure subgroups of order 9), but still satisfies the Norton inequality.

Sakuma's theorem suggested the following definition introduced in [6]. Let G be a finite group. Let T be a generating set of involutions in G which is a union of some conjugacy classes of G. Let V be a real vector space equipped with $(\ ,\ )$ and $\cdot$ satisfying (M1) and (M2). Let

$$\varphi : G \to GL(V) \quad \text{and} \quad \psi : T \to V \setminus \{0\}$$

be a faithful representation and a mapping such that $\psi(t)$ is a Majorana axis for every $t \in T$. Suppose that if $\tau(\psi(t))$ is the Majorana involution defined in (M6) then $\tau(\psi(t)) = \varphi(t)$, and if $g \in G$ conjugates $t_1 \in T$ onto $t_2 \in T$ then $\varphi(g)$ maps $\psi(t_1)$ onto $\psi(t_2)$. Thus we require that $\varphi(G)$ permutes $\psi(T)$ the same way as the conjugation action of G permutes T (since φ is faithful, the latter condition implies that ψ is injective). The Majorana representation of the Monster possesses another important property which we would like to include into the Majorana axiomatic:

(2A) if t_0 and t_1 are involutions in T, whose product is also contained in T, then their respective images a_0 and a_1 generate a subalgebra of type $2A$ and

$$a_\rho = a_0 + a_1 - 8a_0 \cdot a_1$$

is the image of $t_0 t_1$.

Subject to the above conditions (M1)–(M7) together with (2A), the tuple

$$\mathcal{R} = (G, T, V, (\ ,\ ),\ \cdot, \varphi, \psi)$$

is said to be a *Majorana representation* of G. We assume that V is generated by the image of ψ and call $\dim(V)$ the *dimension of* $\mathcal{R}$.

By the above discussion (by Proposition 8.6.2 in [5], to be more specific) the tuple

$$\mathcal{M} = (M, 2A, \Pi, (\ ,\ ),\ \cdot, \vartheta, \eta),$$

where $\eta : t \mapsto a_t$ for $t \in 2A$, is a Majorana representation of the Monster. Furthermore, let $T \subseteq 2A$, let G be the subgroup in M generated by T, and let V be the subspace in Π supporting the subalgebra in the Monster algebra generated by $\{a_t \mid t \in T\}$. Then

$$\mathcal{M}_G = (G, T, V, (\ ,\ )|_V, \cdot|_V, \vartheta|_G, \eta|_T)$$

is a Majorana representation of G, which is said to be *based on an embedding of G into M* (as a $2A$-generated subgroup).

Thus Sakuma's theorems tells us that the dihedral groups possess precisely nine Majorana representations and that all these representations are based on embeddings into the Monster. In [6] Sakuma's result has been expanded to the symmetric group S_4 (four Majorana representations, all based on embeddings into the Monster). In the present paper we handle the alternating group A_5 by proving the following (compare Conjecture 8.8.1. in [5]).

Theorem 1 *The alternating group A_5 of degree 5 possesses precisely two Majorana representations, both are based on embeddings into the Monster. The dimensions of the representations are 20 and 26, while the squared lengths of their identities are 10 and $\frac{72}{7}$, respectively.*

It has been mentioned already that the Norton inequality condition has not been used to prove the above theorem.

The explicit structures of the algebras underlying the representations of A_5 will be recovered within the proof of Theorem 1. These are important subalgebras in the Monster algebra. It was desirable for a long time to unveil their structure but doing so based directly on the Conway–Griess construction in [1,3], is a task no-one has accomplished yet. In this comparison our treatment is the top of transparency, since it does not go beyond calculations in A_5, providing an evident demonstration of the efficiency of the Majorana theory we are building up.

In the final section, we address the dependency issue for the axioms (M1) to (M7) and (2A). As has been emphasised earlier, for all groups G for which the classification of Majorana representations of G has been accomplished so far, Norton's inequality (M2) is not required to complete the classification although it holds in all the resulting algebras. Another question is whether (2A) can be deduced from the other axioms. The answer to this question turns out to be negative. Let us introduce

(2B) there is a pair of involutions in T whose product is also in T and whose images generate a $2B$-type subalgebra.

Theorem 2 *The alternating group A_5 of degree 5 possesses a representation satisfying* (M1), (M3) *to* (M7)*, and* (2B)*. This representation is unique subject to the condition that the images of at least one pair of involutions generate a $3C$-type subalgebra. The dimension of the representation is 21 and the squared length of the identity is 12.*

Although the representation in Theorem 2 cannot possibly be based on an embedding into the Monster, it still satisfies the Norton inequality.

2 Getting started

Let $H \cong A_5$ be the smallest non-abelian simple group isomorphic to the group of even permutations of a 5-element set. For $g \in H$, we denote by $o(g)$ the order of g. Recall that A_5 contains one conjugacy class of elements of order 2 comprising of 15 involutions, one conjugacy class of elements of order 3, containing 20 elements, and two conjugacy classes of elements of order 5, of size 12 each (if f is an element of order 5 then f and f^2 are in different classes).

The following piece of notation deserves a special emphasis.

Definition 2.1 Let $H^{(r)}$ be a set of elements of order r in $H \cong A_5$ containing one representative from every subgroup of order r, so that

$$|H^{(2)}| = 15, \quad |H^{(3)}| = 10, \quad |H^{(5)}| = 6,$$

where $H^{(5)}$ is contained in a single conjugacy class. For $g \in H$ put

$$H_s^{(r)}(g) = \{h \mid h \in H^{(r)},\ o(gh) = s\}.$$

If f and g are elements of order 5 in H then $\sigma_{f,g}$ is defined to be 1 or -1 depending on whether or not f and g are conjugate in H, and σ_f is $\sigma_{f,g}$ for $g \in H^{(5)}$.

We will be dealing with a certain function on H which will not depend on the particular choice of the transversals in Definition 2.1.

The $2A$-generated A_5-subgroups in the Monster and their properties are described in the following.

Proposition 2.2 *There are precisely two conjugacy classes of monomorphisms of $H \cong A_5$ into the Monster subject to the condition that $H^{(2)}$ is mapped into the class of $2A$-involutions. If ζ^{3C} and ζ^{3A} are monomorphisms from different classes so that, for some $h \in H^{(3)}$, $\zeta^{3C}(h)$ is a $3C$-element of the Monster and $\zeta^{3A}(h)$ is a $3A$-element of the Monster, respectively then*

(i) $N_M(\zeta^{3C}(H)) \cong (A_5 \times U_3(8).3).2$ *and* $\dim(C_{\Pi_1}(C_M(\zeta^{3C}(H)))) = 20$*;*
(ii) $N_M(\zeta^{3A}(H)) \cong (A_5 \times A_{12}).2$ *and* $\dim(C_{\Pi_1}(C_M(\zeta^{3A}(H)))) = 26$*.*

Proof The conjugacy classes of the A_5-subgroups in the Monster and their centralizers can be read from Table 3 in [12]. It can be seen from that table that every $2A$-generated A_5-subgroup is fully normalized in the Monster. The dimensions of the centralizers in Π_1 were calculated using [14], via fusion of the character tables of

$$C_M(\zeta^{(3C)}(H)) \cong U_3(8).3 \quad \text{and} \quad C_M(\zeta^{(3A)}(H)) \cong A_{12}$$

into the character table of the Monster (these calculations were performed by Steven Linton and Sergey Shpectorov, respectively). □

By the above proposition, $H \cong A_5$ possesses two Majorana representations $\mathcal{M}_{\zeta^{3C}(H)}$ and $\mathcal{M}_{\zeta^{3A}(H)}$ based on embeddings into the Monster with 21 and 27 being upper bounds for their dimensions. This information has formed the ground for posing Conjecture 8.8.1 in [5].

Now let

$$\mathcal{R} = (H, T, V, (\ ,\ ), \cdot, \varphi, \psi)$$

be an arbitrary Majorana representation of $H \cong A_5$. Since $H^{(2)}$ constitutes a single class of involutions, T must be the whole of $H^{(2)}$. For $t \in H^{(2)}$, let a_t denote the Majorana generator $\psi(t)$. The action of H on V via φ is by naturally conjugating the indices of the Majorana generators, followed by the expansion onto the whole of V via the H-invariance of the algebra product. Let V_1 and V_2 denote the linear spans in V of

$$\{a_t \mid t \in H^{(2)}\} \quad \text{and} \quad \{a_t \cdot a_s \mid t, s \in H^{(2)}\},$$

respectively. Since the Majorana axes are idempotents, V_1 is contained in V_2. Furthermore, since H acts on V via φ and preserves the set of Majorana generators, both V_1 and V_2 are H-submodules in V.

Lemma 2.3 *Let t and s be distinct elements from $H^{(2)}$, let $r = o(ts)$, and let Y be the subalgebra in V generated by a_t and a_s. Then exactly one of the following holds:*

(i) *$r = 2$ and Y is 3-dimensional of type $2A$ linearly spanned by a_t, a_s and a_{ts};*
(ii) *$r = 5$ and Y is 6-dimensional of type $5A$ linearly spanned by $a_t, a_s, a_{sts}, a_{tst}, a_{ststs}$ and w_{ts} (where $w_{ts} = w_{st} = -w_{(ts)^2} = -w_{(ts)^3}$);*
(iii) *$r = 3$ and either*
   (a) *Y is 3-dimensional of type $3C$ linearly spanned by a_t, a_s and a_{tst}; or*
   (b) *Y is 4-dimensional of type $3A$ linearly spanned by a_t, a_s, a_{tst} and u_{ts} (where $u_{ts} = u_{st}$).*

Proof The assertions follow from Proposition 1.1, condition (2A) and Table 2. □

We will say that the order 3 elements in H have type $3C$ or $3A$ depending on whether (a) or (b) holds in Lemma 2.3 (iii). Thus for $t, s \in H^{(2)}$ by writing $ts \in 3A$ we mean that $o(ts) = 3$ and that the subalgebra generated by a_t and a_s is 4-dimensional of type $3A$. This terminology will make a perfect sense *a posteriori*. The group $H \cong A_5$ acts transitively by conjugation on the set of unordered pairs $\{s, t\}$, where $s, t \in H^{(2)}$ and $o(st) = 3$. Hence for all such pairs the subalgebra generated by a_t and a_s is of the same type, and we can say that the algebra is of type $3A$ or $3C$.

Directly by Lemma 2.3 we obtain the following upper bound on $\dim(V_2)$.

Lemma 2.4 *The following assertions hold:*

(i) *if the* 3*-elements in* H *are of type* $3C$*, then* V_2 *is spanned by the* 21*-element set*

$$S^{(3C)} = \{a_t \mid t \in H^{(2)}\} \cup \{w_f \mid f \in H^{(5)}\};$$

(ii) *if the* 3*-elements in* H *are of type* $3A$*, then* V_2 *is spanned by the* 31*-element set*

$$S^{(3A)} = \{a_t \mid t \in H^{(2)}\} \cup \{u_h \mid h \in H^{(3)}\} \cup \{w_f \mid f \in H^{(5)}\}.$$

Lemma 2.5 *Let* $S^{(X)}$ *be the spanning set of* V_2 *as in Lemma* 2.4*, where* $X = 3C$ *or* $3A$ *depending on the type of* 3*-elements in* H*. Then* H *acting on* V_2 *preserves* $S^{(X)}$ *as a whole. Furthermore, the permutation action of* H *on* $S^{(X)}$ *is similar to its action by conjugation on the set of subgroups of order* 2 *and* 5 *in the* $3C$*-case and on its subgroups of order* 2, 3 *and* 5 *in the* $3A$*-case.*

Proof For $g \in H$, the fact that $\varphi(g)$ acts on the Majorana generators by g-conjugation of the corresponding indices implies that

$$a_t^{\varphi(g)} = a_{g^{-1}tg}, \quad u_h^{\varphi(g)} = u_{g^{-1}hg} \quad \text{and} \quad w_f^{\varphi(g)} = w_{g^{-1}fg}. \tag{1}$$

Since u_h and w_f do not depend on the choice between the element in the index and its inverse and since an element of order 5 is not conjugate in H to its second and third powers, the assertion follows. □

At this stage the sense in which the choice of the transversals $H^{(3)}$ and $H^{(5)}$ is irrelevant must become perfectly clear.

Let $U_2^{(X)}$ be a vector space having $S^{(X)}$ as a basis, considered as an H-module in the obvious manner, and let

$$\pi : U_2^{(X)} \to V_2$$

be the corresponding H-projection (of course V_2 also depends on the type X of the 3-elements). By Lemma 2.5, $U_2^{(X)}$ possesses a clear characterization as the direct sum of the permutation modules associated with the conjugation action of H on its sets of subgroups of order 2 and 5 in the $3C$-case, and of order 2, 3 and 5 in the $3A$-case.

3 Some eigenvectors

The following Table 3, which shows the eigenspace decomposition of some of the nontrivial Norton–Sakuma algebras, is essential for our classification of the Majorana representations of A_5. The decompositions are with respect to the adjoint action of a_0 and the 1-eigenvector, which is a_0 itself, is omitted. The eigenvalue properties can be checked directly making use of the multiplication rules in Table 2, while the completeness follows from dimension considerations and from the obvious linear independence of the vectors in Table 3.

Table 3 Eigenspaces in two-Majorana-generated algebras

Type	0	$\frac{1}{2^2}$	$\frac{1}{2^5}$
2A	$a_1 + a_\rho - \frac{1}{2^2}a_0$	$a_1 - a_\rho$	
5A	$-\frac{3}{2^5}a_0 + a_1 + a_{-1} + a_2 + a_{-2}$	$w_\rho + \frac{1}{2^7}(a_1 + a_{-1} - a_2 - a_{-2})$	$a_1 - a_{-1}$
	$w_\rho - \frac{3\cdot 7}{2^{12}}a_0 + \frac{7}{2^6}(a_2 + a_{-2})$		$a_2 - a_{-2}$
3C	$a_1 + a_{-1} - \frac{1}{2^5}a_0$		$a_1 - a_{-1}$
3A	$u_\rho - \frac{2\cdot 5}{3^3}a_0 + \frac{2^5}{3^3}(a_1 + a_{-1})$	$u_\rho - \frac{2^3}{3^2\cdot 5}a_0 - \frac{2^5}{3^2\cdot 5}(a_1 + a_{-1})$	$a_1 - a_{-1}$

Lemma 3.1 *For $t \in H^{(2)}$, let $\{s_1, s_2\}$, $\{h_1, h_2\}$, and $\{f_1, f_2\}$ be the sets $H_2^{(r)}(t)$ for $r = 2, 3,$ and 5, respectively. Then for $i, j \in \{1, 2\}$, each of the following is an eigenvector of (the adjoint action of) a_t on V. The α- and β-vectors are 0- and $\frac{1}{2^2}$-eigenvectors, respectively. For the vectors depending on the type of 3-elements in H the type is indicated by the superscript.*

$$\alpha_t(2) = a_{s_1} + a_{s_2} - \frac{1}{2^2}a_t;$$

$$\alpha_t(f_i, 1) = -\frac{3}{2^5}a_t + \sum_{1\le k\le 4} a_{f_i^{-k}tf_i^k}; \quad \alpha_t(f_i, 2) = w_{f_i} - \frac{3\cdot 7}{2^{12}}a_t + \frac{7}{2^6}\sum_{k=1,4} a_{f_i^{-k}tf_i^k};$$

$$\alpha_t^{(3C)}(h_j) = a_{h_j^{-1}th_j} + a_{h_jth_j^{-1}} - \frac{1}{2^5}a_t;$$

$$\alpha_t^{(3A)}(h_j) = u_{h_j} - \frac{2\cdot 5}{3^3}a_t + \frac{2^5}{3^3}(a_{h_j^{-1}th_j} + a_{h_jth_j^{-1}});$$

$$\alpha_t^{(3C)}(f_i, 3) = -\frac{3}{2^7}a_t + \frac{5}{2^6}(a_{s_1} + a_{s_2}) - \frac{1}{2^6}\sum_{p\in H_3^{(2)}(t)} a_p - \frac{3}{2^7}\sum_{1\le k\le 4} a_{f_{3-i}^{-k}tf_{3-i}^k}$$
$$+\frac{5}{2^7}\sum_{1\le k\le 4} a_{f_i^{-k}tf_i^k} + \sum_{e\in H_3^{(5)}(t)} w_e - \sum_{d\in H_5^{(5)}(t)} w_d;$$

$$\alpha_t^{(3A)}(f_i, 3) = -\frac{3}{2^7}a_t + \frac{11}{2^6}(a_{s_1} + a_{s_2}) - \frac{1}{2^6}\sum_{p\in H_3^{(2)}(t)} a_p + \frac{11}{2^7}\sum_{1\le k\le 4} a_{f_i^{-k}tf_i^k}$$
$$+\frac{3}{2^7}\sum_{1\le k\le 4} a_{f_{3-i}^{-k}tf_{3-i}^k} - \frac{3^3\cdot 5}{2^{11}}\sum_{h\in H_5^{(3)}(t)} u_h + \sum_{e\in H_3^{(5)}(t)} w_e - \sum_{d\in H_5^{(5)}(t)} w_d;$$

$$\alpha_t^{(3C)}(5) = -\frac{3^2}{2^{10}}a_t - \frac{3}{2^6}(a_{s_1} + a_{s_2}) - \frac{5}{2^5}\sum_{p\in H_3^{(2)}(t)} a_p + \frac{23}{2^7}\sum_{q\in H_5^{(2)}(t)} a_q$$
$$-\sum_{e\in H_3^{(5)}(t)} w_e + \sum_{d\in H_5^{(5)}(t)} w_d;$$

$$\alpha_t^{(3A)}(5) = -\frac{3^2}{2^{10}}a_t + \frac{3}{2^6}(a_{s_1} + a_{s_2}) - \frac{7}{2^6}\sum_{p\in H_3^{(2)}(t)} a_p + \frac{5\cdot 7}{2^7}\sum_{q\in H_5^{(2)}(t)} a_q$$

$$-\frac{3^3\cdot 5}{2^{11}}\sum_{h\in H_3^{(3)}(t)\cup H_5^{(3)}(t)} u_h-\sum_{e\in H_3^{(5)}(t)} w_e+\sum_{d\in H_5^{(5)}(t)} w_d;$$

$$\beta_t(2)=a_{s_1}-a_{s_2};\quad \beta_t(f_i)=w_{f_i}-\frac{1}{2^7}(a_{f_i^{-1}tf_i}+a_{f_itf_i^{-1}}-a_{f_i^{-2}tf_i^2}-a_{f_i^{-3}tf_i^3});$$

$$\beta_t^{(3A)}(h_j)=u_{h_j}-\frac{2^3}{3^2\cdot 5}a_t-\frac{2^5}{3^2\cdot 5}(a_{h_j^{-1}th_j}+a_{h_jth_j^{-1}}).$$

Proof The eigenvectors $\alpha_t(2), \alpha_t(f_i, 1), \alpha_t(f_i, 2), \alpha_t^{(X)}(h_j)$ and all the $\frac{1}{2^2}$-ones can be seen inside the subalgebras generated by a_t together with one further Majorana generating axis.

The vector $\alpha_t^{(X)}(f_i, 3)$, $i = 1, 2$, is the product of $\alpha_t(2)$ and $\alpha_t(f_i, 1)$, and $\alpha_t^{(X)}(5)$ is the product of $\alpha_t(f_1, 1)$ and $\alpha_t(f_2, 1)$. All of these products are 0-eigenvectors due to the fusion rules in Table 1. Although the factors do not depend on the type of 3-elements, their product does and hence the superscripts have been applied. All products are linear combinations of terms of the form $a_p \cdot a_q$, for various $p, q \in H^{(2)}$. The terms $a_p \cdot a_q$ can be computed by the multiplication rules in Table 2; the identification of the variables in Table 2 with the elements of $H^{(2)}$ is provided by Lemma 2.3. Carrying out these computation in *GAP* [14] results in the formulas $\alpha_t^{(X)}(f_i, 3)$ and $\alpha_t^{(X)}(5)$ as stated in this lemma. □

We note that we do not claim linear independence of the eigenvectors described in Lemma 3.1.

Remark 3.2 Since the proof of Lemma 3.1 is the first occasion that we use *GAP*, this is the place to comment on the role of computer calculations in this paper. In the $3A$ case, we are working with 31 generators of the algebra (15 a's, 10 u's, and 6 w's), and we try to compute the 31 matrices N_i of size 31×31 describing the algebra products: row j of N_i is the product of the ith and jth generators, expressed as a linear combination of the 31 generators. The product of two arbitrary algebra elements is a linear combination of some rows of the N_i; in extreme cases, all 961 rows occur in the linear combination. We prefer computing algebra products by computer, because hand calculations, although straightforward, are prone to errors.

Later, we have to perform other linear algebra tasks as well. It will turn out that the algebra is 26 dimensional, and we perform a base change to 26 vectors generating the algebra, and 5 vectors that represent 0. We also have to compute the nullspaces of the matrices N_i that describe the multiplications by Majorana axes, and perform Gram–Schmidt orthogonalization for some vectors. All of these tasks are straightforward, and readily accomplished by built-in functions in *GAP*, but would be unpleasant or downright impossible to perform by hand.

The $\frac{1}{2^5}$-eigenvectors can be easily recovered from the Majorana condition (M6). In fact, if $x \in \{a, u, w\}$ and ρ is an element of order 2, 3 (of type $3A$) or 5, and ρ generates a cyclic subgroup in H which is not normalized by $t \in H^{(2)}$, then

$$x_\rho - x_{t\rho t}$$

is a $\frac{1}{2^5}$-eigenvector of a_t [see (1) in the proof of Lemma 2.5]. In this way we obtain 8 and 12 vectors (in the $3C$- and $3A$-cases, respectively) spanning the $\frac{1}{2^5}$-eigenspace of a_t on V_2. These vectors can easily be listed on a demand (similarly to the eigenvectors in Lemma 3.1, the linear independence is not assumed). Notice that 8 and 12 are the dimensions of commutator subspaces of t in the linear spans of $U_2^{(3C)}$ and $U_2^{(3A)}$, respectively.

We will make use of the $\frac{1}{2^5}$-eigenvectors through the following general principle of Majorana calculus.

Lemma 3.3 *Let $v \in V$, $t \in T$, and $a_t = \psi(t)$. Then*

$$a_t \cdot v = \frac{1}{2} a_t \cdot (v + v^{\varphi(t)}) + \frac{1}{2^6}(v - v^{\varphi(t)}).$$

Proof By the definition of Majorana representation, $\varphi(t)$ acts on V as $\tau(a_t)$, therefore by (M6) $v - v^{\varphi(t)}$ is a $\frac{1}{2^5}$-eigenvector (possibly zero) of (the adjoint action of) a_t, and the result follows. □

4 Scalar product on V_2

In this section we reconstruct the scalar product (,) on the subspace V_2. Because of the bilinearity of the scalar product, it suffices to calculate

$$\{(x, y) \mid x, y \in S^{(X)}\}$$

where $S^{(X)}$ is the spanning set of V_2 as in Lemma 2.4 and $X = 3C$ or $X = 3A$ depending on the type of 3-elements in H (i.e., on the type of subalgebras generated by the pairs a_t and a_s of Majorana generators such that the H-product of t and s has order 3). Since both the scalar product and the spanning set $S^{(X)}$ are H-invariant, while the product is symmetric, it is sufficient to calculate the values (x, y) for orbit representatives of H on the set of unordered pairs of vectors from $S^{(X)}$. First we summarise the information on the scalar product implied by the structure of the two-Majorana-generated subalgebras. This information can be read from Table 2.

Lemma 4.1 *The following assertions hold, where $t, s \in H^{(2)}$:*

(i) *if $t = s$ then $(a_t, a_s) = 1$;*
(ii) *if $s \in H_2^{(2)}(t)$ then $(a_t, a_s) = \frac{1}{2^3}$;*
(iii) *if $s \in H_5^{(2)}(t)$ then $(a_t, a_s) = \frac{3}{2^7}$;*
(iv) *if $ts \in 3C$ then $(a_t, a_s) = \frac{1}{2^6}$;*
(v) *if $ts \in 3A$ then $(a_t, a_s) = \frac{13}{2^8}$;*
(vi) *if $h \in 3A$ then $(u_h, u_h) = \frac{2^3}{5}$;*
(vii) *if $f \in H^{(5)}$ then $(w_f, w_f) = \frac{5^3 \cdot 7}{2^{19}}$;*
(viii) *if $h \in H_2^{(3)}(t)$, then $(a_t, u_h) = \frac{1}{2^2}$;*
(ix) *if $f \in H^{(5)}(t)$, then $(a_t, w_f) = 0$.*

The scalar products of the remaining pairs of spanning vectors from $S^{(X)}$ will be recovered from the orthogonality relations for the eigenvectors of a_t with different eigenvalues [these relations hold since by (M1) the scalar and algebra products associate]. Of course it is sufficient to calculate the values of the scalar product for a representative of every H-orbit. Given the action of H on $S^{(X)}$ described in Lemma 2.5 it is an elementary exercise to check that every orbit on pairs of generating vectors is represented either in Lemma 4.1 or in the following Lemma 4.2, which extends via the equalities

$$u_h = u_{h^{-1}}, \quad w_f = -w_{f^2} = -w_{f^3} = w_{f^{-1}}.$$

Lemma 4.2 *The following assertions hold:*

(i) *Let $t \in H^{(2)}$. Then*

(a) *if $f \in H_3^{(5)}(t)$ then $(a_t, w_f) = \frac{7}{2^{13}}$ and $-\frac{7^2}{2^{14}}$ in the $3C$- and $3A$-cases, respectively;*

(b) *if $f \in H_5^{(5)}(t)$ then $(a_t, w_f) = -\frac{7}{2^{13}}$ and $\frac{7^2}{2^{14}}$ in the $3C$- and $3A$-cases, respectively;*

(ii) *if $f, e \in H^{(5)}$ and $e \neq f$, then $(w_f, w_e) = -\frac{5^2 \cdot 7}{2^{19}}$ and $\frac{7 \cdot 29}{2^{19}}$ in the $3C$- and $3A$-cases, respectively;*

(iii) *if $h \in H_r^{(3)}(t)$ then $(a_t, u_h) = \frac{1}{3^2}$ or $\frac{1}{2 \cdot 3^2}$ if $r = 3$ or 5, respectively;*

(iv) *if $f \in H^{(5)}$ and $h \in H^{(3)}$, then $(u_h, w_f) = \frac{5 \cdot 7}{2^9 \cdot 3^2}$ or $-\frac{5 \cdot 7}{2^9 \cdot 3^2}$ if $o(f^{-1}h^{-1}fh) = 3$ or 5, respectively;*

(v) *if $e, h \in H^{(3)}$, $e \neq h$, then $(u_h, u_e) = \frac{2^3 \cdot 17}{3^4 \cdot 5}$ or $\frac{2^4}{3^4 \cdot 5}$ if $o(he) \in \{2, 3\}$ or $o(he) = 5$, respectively.*

Proof To prove (i) we expand the equalities

$$(\alpha_t(2), \beta_t(f_1)) = 0 \quad \text{and} \quad (\alpha_t(f_1, 2), \beta_t(2)) = 0.$$

The first equality implies that the inner products in cases (a) and (b) must differ only by the sign, while the second one provides us with the exact values (depending on the type of 3-elements in H).

Since H acts doubly transitively when conjugating its subgroups of order 5, (ii) can be deduced by expanding the orthogonality relation

$$(\alpha_t(f_1, 2), \beta_t(f_2)) = 0$$

and substituting the necessary scalar product values computed in Lemma 4.1 and in assertion (i) of the current lemma.

From now on within the proof we deal exclusively with the $3A$-case. There are precisely three H-orbits on the set of pairs consisting of a subgroup of order 2 and a subgroup of order 3. The inner products corresponding to one of the orbits is given in Lemma 4.1 (viii). Similarly to the proof of part (i) of this lemma, the pair of equalities

$$(\alpha_t(2), \beta_t^{(3A)}(h_1)) = 0 \quad \text{and} \quad (\alpha_t^{(3A)}(h_1), \beta_t(2)) = 0,$$

computes (u_h, a_m) when $h \in H_5^{(3)}(m)$. After that, we have enough information to expand

$$(\alpha_t(f_1, 1), \beta_t^{(3A)}(h_1)) = 0$$

to obtain (u_h, a_m) when $h \in H_3^{(3)}(m)$.

Now the values in (iv) can be computed in a similar way from the orthogonality relations

$$(\alpha_t^{(3A)}(f_1, 2), \beta_t^{(3A)}(h_1)) = 0, \quad (\alpha_t(f_1, 2), \beta_t^{(3A)}(h_2)) = 0.$$

Finally, the values in (v) are obtained from

$$(\alpha_t^{(3A)}(h_1), \beta_t^{(3A)}(h_2)) = 0$$

when $e \in H_5^{(3)}(h)$ and the last case $e \in H_r^{(3)}(h)$ with $r \in \{2, 3\}$ follows from

$$(\alpha_t^{(3A)}(f_1, 3), \beta_t^{(3A)}(h_1)) = 0.$$

□

Now, as the scalar product on V_2 is fully recovered, we are in a position to calculate the dimension of V_2. In fact, since by (M1) the scalar product is a positive definite bilinear form, a vector $v \in V_2$ is zero if and only if its scalar square is zero. This enables us to determine the linear dependencies between the vectors in $S^{(X)}$, or, equivalently, the kernel of the projective maps

$$\pi : U_2^{(X)} \to V_2.$$

Recall that the transversal $H^{(5)}$ consists of H-conjugate elements of order 5.

Lemma 4.3 *The kernel of the projection $\pi : U_2^{(3C)} \to V_2$ is 1-dimensional spanned by the vector*

$$w := \sum_{f \in H^{(5)}} w_f.$$

Proof The fact that the given sum is in the kernel is immediate by Lemma 4.1 (vii) and Lemma 4.2 (i),(ii). The fact that the linear span of this vector exhausts the kernel has been checked in [14]. □

To describe the kernel of the map $\pi : U_2^{(3A)} \to V_2$ in the $3A$-case, we introduce the following vectors. For $f \in H^{(5)}$, define $A_f := \{h \in H^{(3)} \mid f \in H_2^{(5)}(h) \cup H_2^{(5)}(h^{-1})\}$ and let

$$r(f) = \frac{1}{2^7} \sum_{t \in H_3^{(2)}(f)} a_t - \frac{1}{2^7} \sum_{t \in H_5^{(2)}(f)} a_t + \frac{3^2 \cdot 5}{2^{11}} \sum_{h \in A_f} u_h + w_f.$$

Lemma 4.4 *The kernel of the projection $\pi : U_2^{(3A)} \to V_2$ is 5-dimensional, and it is spanned by the differences $r(f) - r(f')$, for $f, f' \in H^{(5)}$.*

Proof By computation in [14]. □

The vector w defined in Lemma 4.3 is non-zero in the $3A$-case and will play an important role in the subsequent exposition because of the following consequence of Lemmas 4.1, 4.2 and 4.4.

Lemma 4.5 *In the $3A$-case V_2 is 26-dimensional with basis*

$$B^{(3A)} = \{a_t \mid t \in H^{(2)}\} \cup \{u_h \mid h \in H^{(3)}\} \cup \{w\}.$$

Furthermore, for $t \in H^{(2)}$ and $h \in H^{(3)}$ we have

$$(a_t, w) = (u_h, w) = 0, \quad (w, w) = \frac{3^4 \cdot 5 \cdot 7}{2^{17}}.$$

For $f \in H^{(5)}$, we can express w_f in the newly introduced basis $B^{(3A)}$ by writing w_f as a linear combination of $B^{(3A)}$ and vectors of the form $r(f) - r(f')$, for $f' \in H^{(5)}$, and then keeping only the $B^{(3A)}$-part of the linear combination. We obtain

$$w_f = \frac{1}{6} w + \frac{1}{2^7} \left(\sum_{t \in H_5^{(2)}(f)} a_t - \sum_{t \in H_3^{(2)}(f)} a_t \right) + \frac{3^2 \cdot 5}{2^{12}} \left(\sum_{h \in H^{(3)},\, o(f^{-1}h^{-1}fh)=3} u_h - \sum_{h \in H^{(3)},\, o(f^{-1}h^{-1}fh)=5} u_h \right). \tag{2}$$

In the Monster algebra setting, formula (2) was already deduced on p. 300 in [11], modulo a sign discrepancy and a different scaling of basis vectors (in our notation, Norton used the vectors $64a_t, 90u_h, 8192w_f$ as basis).

For $t \in H^{(2)}$ and $s \in H_5^{(2)}(t)$, we can now express $a_t \cdot a_s$ in $B^{(3A)}$ using (2) together the description of the product in terms of the spanning set $S^{(3A)}$ given in Lemma 2.3 (ii) and Table 2.

5 Completing the business

The main assertions in Theorem 1 (apart from the identities to be computed in the next section) are implied by the following

Proposition 5.1 *For a Majorana representation of $H \cong A_5$ the algebra product is closed on V_2 and it is uniquely determined by the type of* 3*-elements.*

The complexity and major details of the proof of Proposition 5.1 depend heavily on the type of 3-elements and so the proof is accomplished in two separate subsections.

5.1 Product closure in the $3C$-case

Throughout the subsection we assume that the 3-elements in H are of type $3C$. Let $V_2(t)$ be the subspace in V_2 spanned a_s taken for all $s \in H^{(2)}$, and by the two w_f for $f \in H_2^{(5)}(t)$. As an immediate consequence of Lemma 2.3 we have the following.

Lemma 5.2 *If $v \in V_2(t)$ then $a_t \cdot v$ is an explicitly computable vector in V_2.* □

Next we compute the product $a_t \cdot w_f$, where

$$f \in H_3^{(5)}(t) \cup H_5^{(5)}(t).$$

For $r = 3$ or 5 the involution t conjugates the two elements in $H_r^{(5)}(t)$. Therefore, by Lemma 3.3 it suffices to calculate the products $a_t \cdot w_r(t)$, where

$$w_r(t) = \sum_{f \in H_r^{(5)}(t)} w_f \quad \text{for } r = 3,\ 5.$$

This will be achieved through the following lemma.

Lemma 5.3 *For $h \in H_2^{(3)}(t)$ let*

$$\alpha = \alpha_t(2) \cdot \alpha_t^{(3C)}(h), \quad \beta = \beta_t(2) \cdot \alpha_t^{(3C)}(h).$$

Then the following assertions hold:

(i) *α and β are* 0*- and $\frac{1}{2^2}$-eigenvectors of a_t, respectively;*
(ii) *there are explicitly computable vectors v_α and v_β which belong to $V_2(t)$ and ε_α, $\varepsilon_\beta \in \{1, -1\}$ such that*

$$\alpha = \varepsilon_\alpha(w_3(t) - w_5(t) + v_\alpha);$$
$$\beta = \varepsilon_\beta(w_3(t) + w_5(t) + v_\beta).$$

Proof The assertion (i) follows from the fusion rules in Table 1, while (ii) is implied by the particular shape of the eigenvectors $\alpha_t(2)$, $\beta_t(2)$ and $\alpha_t^{(3C)}(h)$. □

By Lemmas 5.2 and 5.3(ii), $a_t \cdot v_\alpha$ and $a_t \cdot v_\beta$ are explicitly computable vectors in V_2. Therefore, the equalities

$$a_t \cdot \alpha = 0 \quad \text{and} \quad a_t \cdot \beta = \frac{1}{2^2}\beta$$

enable us to express $a_t \cdot w_3(t)$ and $a_t \cdot w_5(t)$ as linear combinations of vectors in the spanning set $S^{(3C)}$ of V_2. Now it only remains to incur Lemma 3.3 and apply *GAP* to obtain the following explicit version of the product rule.

Lemma 5.4 *Let $t \in H^{(2)}$, let $f \in H_r^{(5)}(t)$ for some $r \in \{3, 5\}$, and suppose that the 3-elements in H are of $3C$-type. Then*

$$a_t \cdot w_f = (-1)^{\frac{r+1}{2}} \frac{7}{2^{13}} a_t - \frac{7}{2^{14}} \sum_{s \in H_5^{(2)}(t)} \sigma_{st}\, a_s + \frac{1}{2^7} \sum_{d \in H_2^{(5)}(t)} w_d + \frac{1}{2^4} \sum_{e \in H_{8-r}^{(5)}(t)} w_e + \frac{5}{2^6} w_f + \frac{3}{2^6} w_{tft}.$$

Here σ_{st} is as in Definition 2.1. □

Next we calculate the product of two distinct w's. Since the conjugation action of H on the set of its order 5 subgroups is doubly transitive, it is sufficient to learn how to multiply w_{f_1} and w_{f_2} for $\{f_1, f_2\} = H_2^{(5)}(t)$. Notice that t is uniquely determined by the pair $\{f_1, f_2\}$ as the only involution in $H^{(2)}$ which inverts both elements in the pair. By Lemmas 5.2 and 5.4, the subspace V_2 is a_t-stable in the sense that it contains the product $a_t \cdot v$ for every $v \in V_2$. We apply the following elementary observation appeared as Lemma 1.8 in [6] and called there a *resurrection principle*.

Lemma 5.5 *Let a be a Majorana axis, and let W be an a-stable subspace of V. For $v \in V$ suppose that*

$$\alpha_v = v + w_\alpha \quad \text{and} \quad \beta_v = v + w_\beta$$

are 0- and $\frac{1}{2^2}$-eigenvectors of a, respectively, for some $w_\alpha, w_\beta \in W$. Then

$$v = -[4a \cdot (w_\alpha - w_\beta) + w_\beta],$$

in particular $v \in W$.

Now we are ready to round up the $3C$-case.

Lemma 5.6 *Let $f_1, f_2 \in H^{(5)}$, $f_1 \neq f_2$ and let $\{t\} = H_2^{(2)}(f_1) \cap H_2^{(2)}(f_2)$. Then*

$$w_{f_1} \cdot w_{f_2} = -\frac{3 \cdot 7}{2^{18}} a_t + \frac{3 \cdot 7}{2^{20}} \sum_{r \in H_2^{(2)}(t)} a_r + \frac{7}{2^{17}} \sum_{q \in H_3^{(2)}(t)} a_q - \frac{7 \cdot 19}{2^{21}} \sum_{s \in H_5^{(2)}(t)} a_s - \frac{7}{2^9}(w_{f_1} + w_{f_2}) - \frac{13 \cdot 19}{2^{14}} \sum_{d \in H_5^{(5)}(t)} w_d - \frac{3 \cdot 67}{2^{14}} \sum_{c \in H_3^{(5)}(t)} w_c.$$

Proof We apply Lemma 5.5 for $W = V_2, a = a_t$,

$$v = w_{f_1} \cdot w_{f_2}, \quad \alpha_v = \alpha_t(f_1, 2) \cdot \alpha_t(f_2, 2), \quad \beta_v = \alpha_t(f_1, 2) \cdot \beta_t(f_2)$$

and perform calculations with [14]. □

The completed case can be summarised as follows.

Proposition 5.7 *Let $\mathcal{H}_{3C} = (H, H^{(2)}, V, (\ ,\ ), \cdot, \varphi, \psi)$ be a Majorana representation of $H \cong A_5$, where $a_t = \psi(t)$ for $t \in H^{(2)}$, and suppose that whenever $t, s \in H^{(2)}$ with $o(ts) = 3$ the subalgebra generated by a_t and a_s is 3-dimensional (of type $3C$). Then*

(i) *V is 20-dimensional spanned by $\{a_t \mid t \in H^{(2)}\} \cup \{w_f \mid f \in H^{(5)}\}$ subject to the relation*

$$\sum_{f \in H^{(5)}} w_f = 0$$

(where $H^{(5)}$ is a 6-element subset of a conjugacy class of elements of order 5 in H, which intersects every cyclic subgroup of that order);

(ii) *the inner product $(\ ,\ )$ is as described in Lemmas* 4.1 and 4.2*;*

(iii) *the algebra product $\cdot$ is as described in Lemmas* 2.3, 5.4 and 5.6.

Thus the isomorphism type of $\mathcal{H}_{3C}$ is uniquely determined and $\mathcal{H}_{3C} \cong \mathcal{M}_{\zeta^{(3C)}(H)}$. □

5.2 Product closure in the $3A$-case

Now we assume that the 3-elements in H are of type $3A$. In this case the *GAP* calculations are more substantial.

Our first objective is to show that V_2 is a_t-stable for $t \in H^{(2)}$.

Lemma 5.8 *Let $V_2(t)$ be the subspace in V_2 spanned by*

$$L = \{a_s \mid s \in H^{(2)}\} \cup H_2^{(3)}(t) \cup \{b - b^{\varphi(t)} \mid b \in B^{(3A)}\}.$$

Then $V_2(t)$ is a 21-dimensional and $a_t \cdot v \in V_2$ for every $v \in V_2(t)$.

Proof The inclusion $a_t \cdot v \in V_2$ for $v \in V_2(t)$ follows from Lemmas 2.3 and 3.3 while the dimension of $V_2(t)$ can be deduced by calculating the rank of the Gram matrix of the spanning set L. □

Notice that $H_3^{(3)}(t)$ and $H_5^{(3)}(t)$ contain four elements each. Let A and B be the t-orbits on $H_3^{(3)}(t)$, let C and D be the t-orbits on $H_5^{(3)}(t)$, for $Z \in \{A, B, C, D\}$ put

$$u_Z = \sum_{d \in Z} u_d,$$

let $K = \{u_A, u_B, u_C, u_D, w\}$, and let $W_2(t)$ be the subspace in V_2 spanned by K.

Lemma 5.9 *The set K is linearly independent and $W_2(t)$ is a complement to $V_2(t)$ in V_2.*

Proof The set $K \cup L$ spans V_2 because L contains all 15 vectors $a_s \in H^{(2)}$, K contains w, and the 8 vectors u_h, for $h \in H^{(3)} \setminus H_2^{(3)}$, can be written as linear combinations in $\{u_h - u_h^{\varphi(t)} \mid h \in H^{(3)}\} \cup \{u_A, u_B, u_C, u_D\}$.

Then, the linear independence of K and the complementation property follows from dimension considerations. □

As a direct consequence of Lemmas 5.8 and 5.9 we have the following.

Lemma 5.10 *The subspace V_2 is a_t-stable if and only if it contains $a_t \cdot k$ for each of the five $k \in K$.* □

For $i \in \{1, 2\}$ and $j \in \{2, 3\}$, consider the a_t-eigenvectors $\alpha_t(f_i, j)$ defined in Lemma 3.1. The relation

$$a_t \cdot \alpha_t(f_i, j) = 0$$

equalizes a linear combination of the vectors in

$$\mathcal{K} = \{a_t \cdot k \mid k \in K\}$$

with a vector from V_2. Solving this system of four equations in the five unknowns in $\mathcal{K}$ by *GAP*, we obtain the following.

Lemma 5.11 *The following assertions hold:*

(i) *each of $a_t \cdot u_A$ and $a_t \cdot u_B$ is contained in V_2;*
(ii) *each of $a_t \cdot u_C$ and $a_t \cdot u_D$ is a scalar multiple of $a_t \cdot w$ plus a vector from V_2.*

The explicit version of Lemma 5.11, combined with $a_t \cdot (u_h - u_h^{\varphi(t)}) = \frac{1}{32}(u_h - u_h^{\varphi(t)})$, gives

Lemma 5.12 *If $h \in H_3^{(3)}(t)$, then*

$$a_t \cdot u_h = \frac{1}{3^2} a_t + \frac{1}{2^6}(5u_h + 3u_{tht} - 4u_{ht} - 4u_{th}).$$ □

It should not be a surprise that the product in the above lemma is exactly as in the second formula in the proof of Lemma 4.16 in [6].

By Lemmas 5.10 and 5.11 (ii) the subspace V_2 is a_t-stable if and only if it contains $a_t \cdot w$. Thus the latter vector will be in the centre of our attention. We start with the following.

Lemma 5.13 *$a_t \cdot w$ is a $\frac{1}{2^2}$-eigenvector of a_t.*

Proof Since $w^{\varphi(t)} = w$, by (M5), (M6) and (M7) we can write

$$w = (a_t, w)\, a_t + w_0 + w_{\frac{1}{2^2}},$$

where w_μ is a μ-eigenvector of a_t. By Lemma 4.5 the vector w is perpendicular to a_t, therefore $a_t \cdot w = \frac{1}{2^2} w_{\frac{1}{2^2}}$ and the assertion follows. □

Lemma 5.14 *Let Q be the subspace in V_2 spanned by the five $\frac{1}{2^2}$-eigenvectors of a_t defined in Lemma 3.1 and let $(a_t \cdot w)_\pi$ be the orthogonal projection of $a_t \cdot w$ into Q. Then*

$$(a_t \cdot w)_\pi = \frac{3}{2^{10}} \sum_{s \in H_2^{(2)}(t)} \sigma_{qs}\, a_s + \frac{3}{2^9} \sum_{r \in H_5^{(2)}} \sigma_{tr}\, a_r + \frac{3^3 \cdot 5}{2^{14}} \sum_{h \in H_5^{(3)}(t)} \sigma_{ht}\, u_h + \frac{1}{2^3} w,$$

where q is an arbitrary element from $H_2^{(3)}(t) = \{h_1, h_2\}$ defined in Lemma 3.1 and σ_f is the function introduced in Definition 2.1.

Proof By Lemmas 4.1 and 4.2, the scalar product on V_2 is known and therefore we can compute an orthogonal basis $\{\beta_1, \dots, \beta_5\}$ for Q by the Gram–Schmidt procedure. Then

$$(a_t \cdot w)_\pi = \sum_{1 \le i \le 5} \frac{(\beta_i, a_t \cdot w)}{(\beta_i, \beta_i)} \beta_i.$$

The quantities $(\beta_i, a_t \cdot w)$ are also computable, since by (M1)

$$(\beta_i, a_t \cdot w) = (a_t \cdot \beta_i, w) = \frac{1}{2^2}(\beta_i, w)$$

and the last expression is a scalar product of two vectors in V_2. Of course, the exact coefficients in $(a_t \cdot w)_\pi$ were computed in *GAP*. □

If we put

$$x_t = a_t \cdot w - (a_t \cdot w)_\pi,$$

then the inclusion $a_t \cdot w \in V_2$ is equivalent to the fact that x_t is the zero vector. In any event let $V_2^+(t)$ be the subspace of V spanned by V_2 together with x_t.

Lemma 5.15 *The subspace $V_2^+(t)$ is a_t-stable.*

Proof The result is an immediate consequence of Lemmas 5.10, 5.11 and 5.13 in view of the observation that x_t is a $\frac{1}{2^2}$-eigenvector of a_t (being the difference of two such eigenvectors). □

Since we have explicit formulas for multiplication by a_t in $V_2^+(t)$, we can compute the eigenspaces of the adjoint action of a_t on $V_2^+(t)$. In particular, we can compute a vector α_t^+ in the 0-eigenspace of a_t that is linearly independent of the vectors described in Lemma 3.1. Let $H_2^{(2)}(t) = \{s_1, s_2\}$, $H_5^{(3)}(t) \cap H_2^{(3)}(s_1) = \{g_1, g_2\}$, and $H_5^{(3)}(t) \cap H_2^{(3)}(s_2) = \{g_3, g_4\}$. Then

$$\alpha_t^+ = -\frac{19}{3^3 \cdot 5} a_t - \frac{2^3}{3 \cdot 5} a_{s_1} + \frac{2^6}{3^3 \cdot 5} \sum_{q \in H_3^{(2)}(t)} a_q + \frac{2^4}{3^2 \cdot 5}(a_{s_1 g_1} + a_{s_1 g_1^{-1}} + a_{s_1 g_2} + a_{s_1 g_2^{-1}})$$
$$- \frac{2^4}{3^2 \cdot 5}(a_{s_2 g_3} + a_{s_2 g_3^{-1}} + a_{s_2 g_4} + a_{s_2 g_4^{-1}}) + (u_{g_1} + u_{g_2}) + \sigma_{t g_1} \frac{2^{12}}{3^3 \cdot 5} x_t$$

is a 0-eigenvector of a_t, and α_t^+ is not in the span of the eigenvectors described in Lemma 3.1.

Lemma 5.16 *Any eigenvector of the adjoint action of a_t on $V_2^+(t)$ in one the following:*

(i) *a scalar multiple of a_t which is a 1-eigenvector;*
(ii) *a linear combination of the ten α-vectors in Lemma 3.1 and the vector α_t^+ which is a 0-eigenvector;*
(iii) *a linear combination of the five β-vectors in Lemma 3.1 and x_t which is a $\frac{1}{2^2}$-eigenvector;*
(iv) *a vector from the 10-dimensional commutator subspace $\{v - v^{\varphi(t)} \mid v \in V_2^+(t)\}$ of the action of t on $V_2^+(t)$, which is a $\frac{1}{2^5}$-eigenvector.*

After x_t will be proved to be the zero vector, Lemma 5.16 will provide us with the eigenspace decomposition of $V_2 = V$ with respect to the adjoint action of a_t (the 0-eigenspace of a_t is of dimension 10; for example, $\alpha_t(f_2, 3)$ can be deleted, and the remaining nine α-vectors of Lemma 3.1, together with α_t^+, give a basis of the 0-eigenspace).

Applying the orthogonality condition of α_t^+ with the 1, $\frac{1}{2^2}$, $\frac{1}{2^5}$-eigenvectors and x_t with the 0, 1, $\frac{1}{2^5}$-eigenvectors, we obtain a description of the scalar product on $V_2^+(t)$ modulo one indeterminate parameter ℓ_t.

Lemma 5.17 *Let $h \in (H_5^{(3)}(t) \setminus H_5^{(3)}(s_1))$, for the element s_1 defined in Lemma 3.1, and let $\ell_t = (x_1, h)$. Then there is a function λ on the set of vectors $b \in B^{(3A)} \cup \{x_t\}$ such that*

(i) $(x_t, b) = \lambda(b)\ell_t$;
(ii) $\lambda(b) = 0$ *if* $b \notin H_5^{(3)}(t) \cup \{w, x_t\}$.

Making use of the known action of a_t on $V_2^+(t)$, we can apply the resurrection principle Lemma 5.5 to express further products as linear combinations of vectors in

$$B^{(3A)}(t) := B^{(3A)} \cup \{x_t\}.$$

As defined in Lemma 3.1, let $\{s_1, s_2\} = H_2^{(2)}(t)$ and $\{h_1, h_2\} = H_2^{(3)}(t)$. Applying Lemma 5.5 first with

$$v = (a_{s_1} + a_{s_2}) \cdot u_{h_1}, \quad \alpha_v = \alpha_t(2) \cdot \alpha_t^{(3A)}(h_1), \quad \beta_v = \alpha_t(2) \cdot \beta_t^{(3A)}(h_1),$$

and then with

$$v = (a_{s_1} - a_{s_2}) \cdot u_{h_1}, \quad \alpha_v = \beta_t(2) \cdot \beta_t^{(3A)}(h_1) - a_t\,(\beta_t(2) \cdot \beta_t^{(3A)}(h_1), a_t),$$
$$\beta_v = \beta_t(2) \cdot \alpha_t^{(3A)}(h_1)$$

we express the product $a_{s_1} \cdot u_{h_1}$ as a linear combination of vectors in $B^{(3A)}(t)$. On the other hand, we introduce the vector x_{s_1} and express $a_{s_1} \cdot u_{h_1}$ as a linear combination of vectors in $B^{(3A)}(s_1) = B^{(3A)} \cup \{x_{s_1}\}$. It turns out that in these two expressions for $a_{s_1} \cdot u_{h_1}$ all vectors from $B^{(3A)}$ have exactly the same coefficients, and the coefficient of x_t in the first expression is equal to the coefficient of x_{s_1} in the second expression. This implies that

$$x_t = x_{s_1}.$$

Lemma 5.18 $x_t = 0$.

Proof Let $h \in (H_5^{(3)}(t) \setminus H_5^{(3)}(s_1))$ be the element used at the definition of ℓ_t. Applying Lemma 5.17 (ii) for x_{s_1}, we obtain $(h, x_{s_1}) = 0$. Hence $x_t = x_{s_1}$ implies $(h, x_t) = \ell_t = 0$ and so, by Lemma 5.17(i), x_t is orthogonal to all vectors in $V_2^+(t)$. In particular, $(x_t, x_t) = 0$ and by (M1) $x_t = 0$. □

By Lemma 5.18, $V_2 = V$ and the action of a_t on V is known. The product $a_t \cdot w = (a_t \cdot w)_\pi$ is given in Lemma 5.14, so the only explicit expression still missing for $a_t \cdot b$, for $b \in B^{(3A)}$, is the following.

For $h \in H_5^{(3)}(t)$, let $\{s_1\} = H_2^{(2)}(t) \setminus H_2^{(2)}(h)$ and $\{s_2\} = H_2^{(2)}(t) \cap H_2^{(2)}(h)$. Moreover, let $H_5^{(3)}(t) \cap H_2^{(3)}(s_1) = \{g_1, g_2\}$, and $H_5^{(3)}(t) \cap H_2^{(3)}(s_2) = \{g_3, g_4\}$ (so $h \in \{g_3, g_4\}$).

Lemma 5.19

$$a_t \cdot u_h = \frac{2}{3^2 \cdot 5} a_t + \frac{1}{3^2 \cdot 5} a_{s_2} - \frac{1}{3^2 \cdot 5} \sum_{q \in H_3^{(2)}(t) \cup \{s_1\}} a_q$$
$$+ \frac{1}{2 \cdot 3^2 \cdot 5} (a_{s_1 g_1} + a_{s_1 g_1^{-1}} + a_{s_1 g_2} + a_{s_1 g_2^{-1}})$$

$$-\frac{1}{2\cdot 3^2\cdot 5}(a_{s_2g_3}+a_{s_2g_3^{-1}}+a_{s_2g_4}+a_{s_2g_4^{-1}})$$
$$+\frac{1}{2^5}\sum_{d\in H_2^{(3)}(t)\cup\{h\}}u_d-\frac{1}{2^6}\sum_{c\in H_2^{(3)}(s_1)}u_c+\sigma_{th}\frac{2^5}{3^3\cdot 5}w.$$

□

There are two H-conjugacy classes on the pairs of 3-element subgroups, and the corresponding product rules are the following.

Lemma 5.20 *Let $h\in H^{(3)}$. Then the following assertions hold:*
(i) *if $g\in H_5^{(3)}(h)$ and $\{t\}=H_2^{(2)}(h)\cap H_2^{(2)}(g)$, then*

$$u_h\cdot u_g=\frac{1}{3^2\cdot 5}(u_h+u_g)+\frac{2^5}{3^4\cdot 5^2}\left(2a_t-\sum_{s\in H_3^{(2)}(t)}a_s\right)$$
$$+\frac{1}{2\cdot 3^2\cdot 5}\left(\sum_{d\in H_5^{(3)}(t)}u_d-\sum_{b\in H_3^{(3)}(t)}u_b\right);$$

(ii) *if $g\in H_2^{(3)}(h)$ and $t=h^{-1}g^{-1}hg$, then*

$$u_h\cdot u_g=\frac{1}{5}(u_h+u_g)-\frac{1}{2\cdot 3^2}(u_{th}+u_{ht})+\frac{2^6}{3^4\cdot 5^2}(2a_t-3a_{hg}-3a_{gh}).$$

Proof (i) We have $\{h,g\}=\{h_1,h_2\}$, for the permutations h_1,h_2 defined in Lemma 3.1. Applying the resurrection principle Lemma 5.5 with

$$v=u_{h_1}\cdot u_{h_2},\quad v_\alpha=\alpha_t^{(3A)}(h_1)\cdot\alpha_t^{(3A)}(h_2),\quad v_\beta=\alpha_t^{(3A)}(h_1)\cdot\beta_t^{(3A)}(h_2),$$

a little computation in *GAP* gives the stated result.
(ii) There are 30 pairs $\{h,g\}\subset H^{(3)}$ corresponding to the second H-conjugacy class of pairs of order 3 subgroups, and we write a system of linear equations for the 30 unknowns $u_h\cdot u_g$ the following way. For the 15 elements $t\in H^{(2)}$, g_1,g_2 as defined in the definition of α_t^+, and $i=1,2$, applying Lemma 5.5 with

$$v=u_{h_i}\cdot(u_{g_1}+u_{g_2}),\quad v_\alpha=\alpha_t^{(3A)}(h_i)\cdot\alpha_t^+,\quad v_\beta=\beta_t^{(3A)}(h_i)\cdot\alpha_t^+,$$

we express $u_{h_i}\cdot u_{g_1}+u_{h_i}u_{g_2}$ as a linear combination of vectors in V. Yet the system of 30 linear equations obtained in this way is still insufficient to find all the 30 unknowns $u_h\cdot u_g$ in question, since the rank of the system turns out to be only 22 (as checked by *GAP*). In order to close the gap, we write further 30 equations using α_t^- instead of α_t^+ (here α_t^- is defined by exchanging the roles of s_1 and s_2 in the definition of α_t^+: an arbitrary choice was made while defining α_t^+). The total rank finally becomes 30, and solving the system in *GAP* gives all the desired products $u_h\cdot u_g$. □

Again the product expression in Lemma 5.20 (ii) is the same as the one in the proof of Lemma 4.16 in [6].

The final two product rules involve the vector w.

Lemma 5.21 *If $h \in H^{(3)}$ then*

$$u_h \cdot w = \frac{1}{2^5 \cdot 5} \sum_{t \in H_5^{(2)}(h)} \sigma_{th}\, a_t + \frac{1}{3 \cdot 5} w;$$

$$w \cdot w = \frac{3^3 \cdot 5}{2^{16}} \sum_{t \in H^{(2)}} a_t - \frac{3^5 \cdot 5}{2^{21}} \sum_{h \in H^{(3)}} u_h.$$

Proof Since almost all products in the algebra are already known, these proofs are easy applications of Lemma 5.5. □

The completed case is summarized in the following.

Proposition 5.22 *Let $\mathcal{H}_{3A} = (H, H^{(2)}, V, (\ ,\ ), \cdot, \varphi, \psi)$ be a Majorana representation of $H \cong A_5$, where $a_t = \psi(t)$ for $t \in H^{(2)}$, and suppose that whenever $t, s \in H^{(2)}$ with $o(ts) = 3$ the subalgebra generated by a_t and a_s is 4-dimensional (of type $3A$). Then*

(i) *V is 26-dimensional with basis $\{a_t \mid t \in H^{(2)}\} \cup \{u_h \mid h \in H^{(3)}\} \cup \{w\}$, where $w = \sum_{f \in H^{(5)}} w_f$ with $H^{(5)}$ being a 6-element subset of a conjugacy class of elements of order 5 in H, which intersects every cyclic subgroup of that order;*
(ii) *the inner product $(\ ,\ )$ is as described in Lemmas* 4.1 and 4.2*;*
(iii) *the algebra product $\cdot$ is as described in Lemmas* 2.3, 5.12, 5.14, 5.19, 5.20 and 5.21.

Thus the isomorphism type of $\mathcal{H}_{3A}$ is uniquely determined and $\mathcal{H}_{3A} \cong \mathcal{M}_{\zeta^{(3A)}(H)}$. □

6 Identities

In this section we determine the identities in the algebras supporting the Majorana representations $\mathcal{H}_{3C}$ and $\mathcal{H}_{3A}$ of A_5 and calculate their scalar squares.

Lemma 6.1 *Let $(V, (\ ,\ ), \cdot)$ be a triple satisfying* (M1)*, let ι be an identity, and let d be an idempotent (so that $d \cdot d = d$ and $\iota \cdot v = v$ for every $v \in V$). Then*

(i) *the scalar product of ι and d is equal to the scalar square of d;*
(ii) *ι is the only identity in V.*

Proof Assertion (i) is implied by the following sequence of equalities

$$(\iota, d) = (\iota, d \cdot d) = (\iota \cdot d, d) = (d, d).$$

Let ι' be another identity, and put $\delta = \iota' - \iota$. Then δ annihilates every vector of V, and since $v = v \cdot \iota$ for every $v \in V$, we have

$$(\delta, v) = (\delta, v \cdot \iota) = (\delta \cdot \iota, v) = (0, v) = 0,$$

which shows that δ is perpendicular to every vector from V. Since the scalar product is a positive definite form on V, this means that δ is the zero vector, $\iota' = \iota$, and (ii) follows. □

Proposition 6.2 *If*

$$a = \sum_{t \in H^{(2)}} a_t, \quad \text{and} \quad u = \sum_{h \in H^{(3)}} u_h,$$

then

(i) $\iota_{3C} = \frac{2}{3}a$ *is the unique identity of* $\mathcal{H}_{3C}$*, and* $(\iota_{3C}, \iota_{3C}) = 10$*;*
(ii) $\iota_{3A} = \frac{16}{35}a + \frac{3}{14}u$ *is the unique identity of* $\mathcal{H}_{3A}$*, and* $(\iota_{3A}, \iota_{3A}) = \frac{72}{7}$.

Proof For symmetry reasons, we are seeking the identities of the algebras in the form

$$\iota_{3C} = x\,a, \quad \text{and} \quad \iota_{3A} = y\,a + z\,u + r\,w,$$

for some real parameters x, y, z and r (it should probably be emphasised that a depends on the type X of 3-elements in H and note that in the 3C-case, the vector w is equal to 0).

The system of equations $\iota_{3C} \cdot a_t = a_t$ for $t \in H^{(2)}$, $\iota_{3C} \cdot w_f = w_f$ for $f \in H^{(5)}$ has the unique solution $x = \frac{2}{3}$. Similarly, the equations $\iota_{3A} \cdot a_t = a_t$ for $t \in H^{(2)}$, $\iota_{3A} \cdot u_h = u_h$ for $h \in H^{(3)}$, $\iota_{3A} \cdot w = w$ has the unique solution $y = \frac{16}{35}$, $z = \frac{3}{14}$, $r = 0$. So, we found identities ι_X in the algebras and, by Lemma 6.1 (ii), ι_X is the only identity of $\mathcal{H}_X$.

The scalar squares of the identities can be computed using Lemmas 4.1 and 4.2. (Using Lemma 6.1 (i), the scalar square computations can also be done by hand.) □

7 The axiom (2A)

In this section we prove Theorem 2. This theorem demonstrates that the axiom (2A) does not follow from the other axioms of Majorana representations.

In this section, let

$$\mathcal{R} = (H, T, V, (\ ,\ ), \cdot, \varphi, \psi)$$

be an arbitrary representation of $H \cong A_5$ satisfying (M1), (M3)–(M7), (2B), and the property that there exist $s, t \in H^{(2)}$ such that $o(st) = 3$ and a_t and a_s generate an algebra of type $3C$.

We shall prove Theorem 2 by mimicking the argument used for the proof of Theorem 1. Let V_1 and V_2 denote the linear spans in V of

$$\{a_t \mid t \in H^{(2)}\} \quad \text{and} \quad \{a_t \cdot a_s \mid t, s \in H^{(2)}\},$$

respectively. We have $V_1 \leq V_2$, and both V_1 and V_2 are H-submodules in V.

Lemma 7.1 *Let t and s be distinct elements in $H^{(2)}$, let $r = o(ts)$, and let Y be the subalgebra in V generated by a_t and a_s. Then exactly one of the following holds:*

(i) *$r = 2$ and Y is 2-dimensional of type $2B$ linearly spanned by a_t and a_s;*
(ii) *$r = 5$ and Y is 6-dimensional of type $5A$ linearly spanned by $a_t, a_s, a_{sts}, a_{tst}, a_{ststs}$ and w_{ts} (where $w_{ts} = w_{st} = -w_{(ts)^2} = -w_{(ts)^3}$);*
(iii) *$r = 3$ and Y is 3-dimensional of type $3C$ linearly spanned by a_t, a_s and a_{tst}.*

Proof There are 15 unordered pairs of commuting involutions in H, and H acts transitively by conjugation on the set consisting of these 15 pairs. Hence, for all $t, s \in H^{(2)}$ with $o(ts) = 2$, the algebra generated by a_t and a_s is of the same type. Since $\mathcal{R}$ satisfies (2B), this common type must be $2B$ and part (i) holds. Part (ii) follows from Proposition 1.1.

The proof of (iii) is similar to (i). There are 30 unordered pairs $\{s, t\}$ of involutions in H with $o(st) = 3$. The group H acts transitively by conjugation on the set consisting of these 30 pairs, so for all pairs $\{s, t\}$, a_s and a_t generate an algebra of the same type. It is a hypothesis of Theorem 2 that this common type is $3C$. □

As a consequence, we obtain

Lemma 7.2 (i) *The vector space V_2 is spanned by the* 21*-element set*

$$S^{(2B,3C)} = \{a_t \mid t \in H^{(2)}\} \cup \{w_f \mid f \in H^{(5)}\}.$$

(ii) *The H-action on V_2 permutes the set $S^{(2B,3C)}$, with the same permutation action as the H-action by conjugation on the set of subgroups of order* 2 *and* 5 *in H.* □

For $t \in H^{(2)}$, let $H_2^{(2)}(t) = \{s_1, s_2\}$, $H_2^{(3)}(t) = \{h_1, h_2\}$, and $H_2^{(5)}(t) = \{f_1, f_2\}$. Moreover, let $G_1 = H_5^{(3)}(t) \cap H_2^{(3)}(s_1)$ and $G_2 = H_5^{(3)}(t) \cap H_2^{(3)}(s_2)$.

Lemma 7.3 *Let $t \in H^{(2)}$, and let s_i, h_i, f_i, G_i as defined above. Then, for $i \in \{1, 2\}$, each of the following is an eigenvector of (the adjoint action of) a_t on V. The α- and β-vectors are* 0*- and $\frac{1}{2^2}$-eigenvectors, respectively.*

$$\alpha_t(s_i) = a_{s_i};$$

$$\alpha_t(f_i, 1) = -\frac{3}{2^5}a_t + \sum_{1\le k\le 4} a_{f_i^{-k}tf_i^k}; \quad \alpha_t(f_i, 2) = w_{f_i} - \frac{3\cdot 7}{2^{12}}a_t + \frac{7}{2^6}\sum_{k=1,4} a_{f_i^{-k}tf_i^k};$$

$$\alpha_t(h_i) = a_{h_i^{-1}th_i} + a_{h_ith_i^{-1}} - \frac{1}{2^5}a_t;$$

$$\alpha_t(s_i, h_i) = \frac{3}{2^6}a_{s_i} + \frac{3}{2^7}(a_{h_i^{-1}th_i} + a_{h_ith_i^{-1}}) - \frac{1}{2^7}(a_{h_{3-i}^{-1}th_{3-i}} + a_{h_{3-i}th_{3-i}^{-1}})$$
$$-\frac{1}{2^7}\sum_{g\in G_{3-i}} (a_{s_{3-i}g} + a_{s_{3-i}g^{-1}}) + w_{s_{3-i}h_i} + w_{s_{3-i}h_i^{-1}};$$

$$\beta_t(f_i) = w_{f_i} - \frac{1}{2^7}(a_{f_i^{-1}tf_i} + a_{f_itf_i^{-1}} - a_{f_i^{-2}tf_i^2} - a_{f_i^{-3}tf_i^3}). \quad \square$$

Proof The eigenvectors $\alpha_t(s_i)$, $\alpha_t(f_i, 1)$, $\alpha_t(f_i, 2)$, $\alpha_t(h_i)$, $\beta_t(f_i)$ can be seen inside the subalgebras generated by a_t together with one further Majorana generating axis.

The vector $\alpha_t(s_i, h_i)$ is the product of $\alpha_t(s_i)$ and $\alpha_t(h_i)$, expanded as a linear combination of $S^{(2B,3C)}$ using the product rules in Table 2. The identification between $H^{(2)}$ and the variables in Table 2 is provided by Lemma 7.1. □

For each $t \in H^{(2)}$, the w_f terms appearing in the vectors $\alpha_t(s_i, h_i)$ are

$$\sum_{f\in H_5^{(5)}(t)} w_f \quad \text{and} \quad -\sum_{f\in H_3^{(5)}(t)} w_f.$$

(Which of these two expressions appears in which $\alpha_t(s_i, h_i)$ depends on the arbitrary choice between s_1 and s_2; the choice between h_1 and h_2 does not change these two expressions.) Hence, using that the two elements in $H_r^{(5)}(t)$ are t-conjugates of each other for $r \in \{3, 5\}$ and $a_t \cdot \alpha_t(s_i, h_i) = 0$, Lemma 3.3 gives the products $a_t \cdot w_f$ for all $f \in H_3^{(5)}(t) \cup H_5^{(5)}(t)$.

The still missing products are $w_f \cdot w_{f'}$, for $f, f' \in H^{(5)}$, $f \neq f'$. Each such pair $\{f, f'\}$ occurs as $\{f_1, f_2\}$ for some $t \in H^{(2)}$, and their product can be computed by applying Lemma 5.5 with $W = V_2$, $a = a_t$,

$$v = w_{f_1} \cdot w_{f_2}, \quad \alpha_v = \alpha_t(f_1, 2) \cdot \alpha_t(f_2, 2), \quad \beta_v = \alpha_t(f_1, 2) \cdot \beta_t(f_2).$$

The explicit formulas are the following.

Lemma 7.4 *For $t \in H^{(2)}$ and $f \in H_r^{(5)}(t)$ with $r \in \{3, 5\}$,*

$$a_t \cdot w_f = (-1)^{\frac{r-1}{2}} \frac{1}{2^{13}} a_t + \sum_{q \in H_5^{(2)}(t)} \sigma_{qt} \frac{1}{2^{14}} a_q + \sum_{p \in H_2^{(5)}(t)} \frac{1}{2^7} w_p + \frac{1}{2^6} w_f - \frac{1}{2^6} w_{tft}.$$

□

Lemma 7.5 *For $t \in H^{(2)}$ and $f, g \in H_2^{(5)}(t)$,*

$$w_f \cdot w_g = \frac{3}{2^{18}} a_t - \frac{3^3}{2^{20}} \sum_{s \in H_2^{(2)}(t)} a_s + \frac{3^3}{2^{21}} \sum_{r \in H_5^{(2)}(t)} a_r - \frac{5^2}{2^{14}} \sum_{p \in H_3^{(5)}(t)} w_p + \frac{5^2}{2^{14}} \sum_{q \in H_5^{(5)}(t)} w_q.$$

□

Since the set V_2 is closed for the algebra product, we have $V = V_2$.

Lemma 7.6 *For $t, s \in H^{(2)}$ the following hold:*

(i) *if $t = s$ then $(a_t, a_s) = 1$;*
(ii) *if $s \in H_2^{(2)}(t)$ then $(a_t, a_s) = 0$;*
(iii) *if $s \in H_3^{(2)}(t)$ then $(a_t, a_s) = \frac{1}{2^6}$;*
(iv) *if $s \in H_5^{(2)}(t)$ then $(a_t, a_s) = \frac{3}{2^7}$;*
(v) *if $f \in H^{(5)}$ then $(w_f, w_f) = \frac{5^3 \cdot 7}{2^{19}}$;*
(vi) *if $f \in H_2^{(5)}(t)$, then $(a_t, w_f) = 0$;*
(vii) *if $f \in H_3^{(5)}(t)$, then $(a_t, w_f) = -\frac{1}{2^{13}}$;*
(viii) *if $f \in H_5^{(5)}(t)$, then $(a_t, w_f) = \frac{1}{2^{13}}$;*
(ix) *if $f, g \in H_2^{(5)}(t)$, $f \neq g$ then $(w_f, w_g) = \frac{3 \cdot 11}{2^{19}}$.*

Proof Parts (i)–(vi) can be read out from Table 2. Parts (vii) and (viii) follow from the orthogonality of vectors $\alpha_t(s_i)$ and $\beta_t(f_j)$, $i, j \in \{1, 2\}$, in Lemma 7.3, utilizing the already known (i)–(vi). Finally, (ix) can be obtained from the orthogonality of $\alpha_t(f_1, 2)$ and $\beta_t(f_2)$ in Lemma 7.3. □

Since the scalar products in V are known, we can check in *GAP* that there are no vectors in V with scalar square 0, i.e., the dimension of V is 21. For $t \in H^{(2)}$, the dimensions of the 0, $\frac{1}{2^2}$, $\frac{1}{2^5}$-eigenspaces are 10, 2, and 8, respectively. The eigenspaces are spanned by the ten α-vectors listed in Lemma 7.3, the two β-vectors in that lemma, and the set of vectors $\{v - v^{\varphi(t)} \mid v \in S^{(2B,3C)}\}$, respectively.

Finally, we compute the identity of the algebra. Let

$$a = \sum_{t \in H^{(2)}} a_t, \quad \text{and} \quad w = \sum_{f \in H^{(5)}} w_f.$$

Lemma 7.7 *The identity of the algebra is $\iota_{\mathcal{R}} = \frac{4}{5} a$, and $(\iota_{\mathcal{R}}, \iota_{\mathcal{R}}) = 12$.*

Proof For symmetry reasons, we seek the identity of the algebra in the form $\iota_{\mathcal{R}} = xa + yw$, for some real numbers x, y. The linear system of equations $\iota_{\mathcal{R}} \cdot v = v$, for $v \in S^{(2B,3C)}$, has the unique solution $x = \frac{4}{5}$, $y = 0$. By Lemma 6.1(ii), $\iota_{\mathcal{R}}$ is the unique identity of the algebra. □

Since the representation $\mathcal{R}$ we have constructed is not based on an embedding into the Monster, we have to check that the axioms (M1), (M3)–(M7) hold. The eigenvalues of the Gram matrix of the algebra are $\frac{5}{2^2}$ with multiplicity 1, $\frac{5\cdot 13}{2^6}$ with multiplicity 5, $\frac{3\cdot 5}{2^4}$ with multiplicity 4, $\frac{5\cdot 13}{2^{15}}$ with multiplicity 1, and the two roots of

$$x^2 - \frac{258469}{2^{18}}x + \frac{3\cdot 5\cdot 13\cdot 17}{2^{21}},$$

each with multiplicity 5. We also computed that the relation $(u, v\cdot w) = (u\cdot v, w)$ holds for all $u, v, w \in S^{(2B,3C)}$; hence, by the linearity of the algebra and scalar products, the relation holds for all $u, v, w \in V$. Thus (M1) is satisfied.

The axioms (M3)–(M5) follow from our construction. We checked the validity of (M6) and (M7) by verifying that for each $t \in H^{(2)}$, the eigenvectors of a_t satisfy the fusion rules of Table 1.

We prove that $\mathcal{R}$ satisfies (M2) using the following lemma, which is based on ideas from pp. 530–531 of [1].

Lemma 7.8 *Let V be an n-dimensional algebra with commutative algebra product $\cdot$ and scalar product $(\ ,\ )$. Let $\{v_i \mid 1 \le i \le n\}$ be a basis of V, and define an $(n^2\times n^2)$-dimensional matrix $B = (b_{ij,k\ell})$ in the following way. The rows and columns are indexed by the ordered pairs (i, j) for $1 \le i, j \le n$, and*

$$b_{ij,kl} = (v_i\cdot v_k, v_j\cdot v_\ell) - (v_j\cdot v_k, v_i\cdot v_\ell).$$

If B is positive semidefinite then V satisfies Norton's inequality (M2).

Proof For $x, y \in V$, write x and y as linear combinations

$$x = \sum_{i=1}^{n} x_i v_i \quad\text{and}\quad y = \sum_{j=1}^{n} y_j v_j$$

and form the n^2-long vector z with entries $x_i y_j$. In this vector z, the coordinate $x_i y_j$ is in the position indexed by (i, j) in the definition of B. Then the inequality $(x\cdot x, y\cdot y) - (x\cdot y, x\cdot y) \ge 0$ is equivalent to $zBz^T \ge 0$. Hence the positive semidefinity of B implies that (M2) holds in V. □

Igor Faradzev computed that, for the 21-dimensional algebra of this section, the matrix B constructed from $S^{(2B,3C)}$ is positive semidefinite, with a 395-dimensional nullspace. Hence (M2) holds in $\mathcal{R}$.

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